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SUPERSTABLE OPERATORS ON BANACH SPACES*

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ABSTRACT

It has been shown by W. Arendt - C.J.K. Batty and Yu.I. Lyubich - V.Q. Phong that the powers of a linear contraction on a reflexive Banach space converge strongly to zero if the boundary spectrum is countable and contains no eigenvalues. In this paper we characterize the countability of the boundary spectrum through a stronger convergence property in terms of ultrapower extensions.

1. Stable operators

The powers T^n of a bounded linear operator T on a Banach space E might exhibit quite different behavior as n tends to infinity ranging from very regular to very irregular ('chaotic'). An appropriate notion for describing regular (and well understood (see [Na])) behavior seems to be the following.

Definition 1.1: A bounded linear operator $T \in \mathcal{L}(E)$, E a Banach space, is called **stable** if $\{T^n x : n \in \mathbb{N}\}$ is relatively compact for every $x \in E$, i.e., $\{T^n : n \in \mathbb{N}\}$ is relatively compact for the strong operator topology on $\mathcal{L}(E)$. ∎

Two contrasting examples show the range of this notion.

Example 1.2: (a) Let G be any compact group. Fix an element $g \in G$ and define on $E := C(G)$ the operator $T_g f(h) := f(hg)$ for all $h \in G$ and $f \in C(G)$. Then T_g is called the **(right) rotation operator** induced by g. Since $g \mapsto T_g$ is continuous from G into $\mathcal{L}(E)$ endowed with the strong operator topology and

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since $T_{g^n} = (T_g)^n$ it follows that $\{T_g^n : n \in \mathbb{N}\}$ is relatively compact and hence T_g is a stable operator.

(b) On the other hand every $T \in \mathcal{L}(E)$ satisfying $\lim_{n\to\infty} \|T^n x\| = 0$ for every $x \in E$ is stable. A concrete example is the shift $T : (z_n) \mapsto (z_{n+1})$ on l^p, $1 \le p < \infty$. ∎

In many cases, e.g. for power bounded operators on reflexive Banach spaces, one already knows that $\{T^n : n \in \mathbb{N}\}$ is at least relatively compact for the weak operator topology. Then the Glicksberg–deLeeuw splitting theorem can be applied yielding very useful information. We refer to [Kr], 2.4.4, 2.4.5, but briefly state the result in the form we will make use of it.

PROPOSITION 1.3: *Let T be a power bounded linear operator on a reflexive Banach space E. Then E is the direct sum of the two closed T-invariant subspaces*

$$E_r := \overline{\text{lin}}\{x \in E : \text{ there exists } \lambda \in \mathbb{C}, |\lambda| = 1, \text{ such that } Tx = \lambda x\}$$

and

$$E_{ws} := \{y \in E : 0 \text{ is a weak accumulation point of } \{T^n y : n \in \mathbb{N}\}\}.$$

Moreover, the weak operator closure of $\{T^n : n \in \mathbb{N}\}$ restricted to E_r is a group which is compact in the strong operator topology on $\mathcal{L}(E_r)$.

From this splitting theorem it follows that the orbits $\{T^n x : n \in \mathbb{N}\}$ are already relatively norm compact for every $x \in E_r$. Hence we only have to improve the convergence towards zero for $y \in E_{ws}$. This is stated as a corollary.

COROLLARY 1.4: *Let T be a power bounded linear operator on a reflexive Banach space E. Then T is stable if and only if E_{ws} coincides with $E_s := \{y \in E : \lim_{n\to\infty} \|T^n y\| = 0\}$.*

It is therefore important to have good criteria for the strong convergence to zero of the powers T^n. We concentrate on spectral conditions using the following 'rule of thumb': the thinner the part of the spectrum $\sigma(T)$ situated on the unit circle $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ the better the convergence to zero of T^n.

In fact, Arendt and Batty [A-B] and independently Lyubich and Phong [L-P1], [L-P2] found the following beautiful result which we again only state for reflexive Banach spaces.

THEOREM 1.5: *Let T be a power bounded linear operator on a reflexive Banach space E. If*

$$(ABLP) \qquad \Lambda := \sigma(T) \cap \Gamma \text{ is countable,}$$

then T and its adjoint T' is stable.

Proof: We perform the Glicksberg-deLeeuw splitting and obtain that for the restriction T_{ws} of T to E_{ws} the boundary spectrum $\sigma(T_{ws}) \cap \Gamma$ is still countable but T_{ws} has no eigenvalues on Γ. Therefore Theorem 5.1 in [A-B] or Theorem 2 in [L-P2] implies that $\lim_{n\to\infty} \|T^n x\| = 0$ for every $x \in E_{ws}$. The stability on E_r is clear from the characterization of E_r in Proposition 1.3. The stability of T' follows by the same arguments since $\sigma(T') \cap \Gamma = \sigma(T) \cap \Gamma$. ∎

We have seen that the spectral condition $(ABLP)$ implies stability. But even in the absence of point spectrum this is not necessary.

Example 1.6: Let $E := L^2(D, m)$ for $D := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and m the Lebesgue measure. Take the multiplication operator $Mf(\lambda) := \lambda \cdot f(\lambda)$ for $\lambda \in D$, $f \in E$. Then $\sigma(M) = D$, the point spectrum $P\sigma(M)$ is empty but $\lim_{n\to\infty} \|M^n f\| = 0$ for every $f \in E$, *i.e.*, M (and M') is stable but $\sigma(M) \cap \Gamma = \Gamma$. ∎

While condition $(ABLP)$ is not necessary it is 'optimal' in the sense explained by the following example.

Example 1.7: Let Λ be a closed uncountable subset of Γ. By [Se], 19.7.6, 8.5.5 there exists a diffuse probability measure μ whose support is contained in Λ. On $E := L^2(\Lambda, \mu)$ we take again the multiplication operator $Mf(\lambda) := \lambda \cdot f(\lambda)$ for $\lambda \in \Lambda$, $f \in E$, and obtain that M is not stable. In fact, $P\sigma(M) = \emptyset$ since μ is diffuse. Therefore $E = E_{ws}$ and it follows from Corollary 1.4 that M is stable if and only if $\lim_{n\to\infty} \|M^n f\| = 0$. But M is an isometry. ∎

These observations and examples lead to the following question which is to be answered later.

PROBLEM 1.8: *Given a power bounded linear operator T on a reflexive Banach space E, which convergence property of T (or some relative of T) is equivalent to the spectral condition*

$$(ABLP) \qquad \sigma(T) \cap \Gamma \text{ is countable?}$$

2. Superstable operators

We propose to answer Problem 1.8 using ultrapower techniques and recall briefly the basic definitions. For more details and a wealth of interesting results we refer to [He], [H-M], [Si] or [St].

Definition 2.1: (a) Let E be a Banach space and $\mathcal{U}$ a (free) ultrafilter on $\mathbb{N}$. Consider the Banach space $l^\infty(E)$ of all bounded sequences in E and its closed subspace $c_{\mathcal{U}}(E)$ of all bounded sequences converging to zero along $\mathcal{U}$. We call the quotient space $E_{\mathcal{U}} := l^\infty(E)/c_{\mathcal{U}}(E)$ the ($\mathcal{U}$-)**ultrapower** of E and denote its elements by $\hat{x} = (x_n)_{n\in\mathbb{N}} + c_{\mathcal{U}}(E)$ (or simply $\hat{x} = (x_n)_{n\in\mathbb{N}}$). Recall that $\|\hat{x}\| = \mathcal{U} - \lim \|x_n\|$ for every $\hat{x} = (x_n)_{n\in\mathbb{N}} \in E_{\mathcal{U}}$.

(b) Let $T \in \mathcal{L}(E)$ and define an operator $T_{\mathcal{U}} \in \mathcal{L}(E_{\mathcal{U}})$ by $T_{\mathcal{U}}\hat{x} := (Tx_n)_{n\in\mathbb{N}} + c_{\mathcal{U}}(E)$ for every $\hat{x} = (x_n)_{n\in\mathbb{N}} \in E_{\mathcal{U}}$. ∎

Since E is isometrically imbedded in $E_{\mathcal{U}}$ (via $x \mapsto (x, x, ...)$) it follows that $T_{\mathcal{U}}$ is an extension of T to $E_{\mathcal{U}}$, clearly preserving many properties of the original operator. In particular, since $T \mapsto T_{\mathcal{U}}$ is an algebra homomorphism from $\mathcal{L}(E)$ into $\mathcal{L}(E_{\mathcal{U}})$, invertibility of T and therefore the spectrum $\sigma(T)$ of T coincides with $\sigma(T_{\mathcal{U}})$. Moreover, and this is one of the main advantages of ultrapower techniques for our purpose, the approximate point spectrum

$$A\sigma(T) := \{\lambda \in \mathbb{C}:\ \text{there exist } x_n \in E,\ \|x_n\| = 1 \\ \text{such that } \lim_{n\to\infty} \|Tx_n - \lambda x_n\| = 0\}$$

is converted into the point spectrum $P\sigma(T_{\mathcal{U}})$ of $T_{\mathcal{U}}$. Since the topological boundary of $\sigma(T)$ is always contained in $A\sigma(T)$ (see [Do], Thm.1.16) it follows that for a power bounded operator the boundary spectrum $\Lambda := \sigma(T) \cap \Gamma$ is contained in $P\sigma(T_{\mathcal{U}})$. We restate these facts in the following proposition, but refer to [Sc], Ch.V, §1 for proofs and numerous applications.

PROPOSITION 2.2: *Let $T \in \mathcal{L}(E)$ and $T_{\mathcal{U}}$ be its canonical extension to some ultrapower $E_{\mathcal{U}}$. Then $\sigma(T) = \sigma(T_{\mathcal{U}})$ and $A\sigma(T) = P\sigma(T_{\mathcal{U}})$.*

The coincidence of $\sigma(T)$ and $\sigma(T_{\mathcal{U}})$ has the consequence that $T_{\mathcal{U}}$ satisfies $(ABLP)$ if (and only if) T satisfies $(ABLP)$. So, we obtain stability of $T_{\mathcal{U}}$ from Theorem 1.5 whenever T (and hence $T_{\mathcal{U}}$) is power bounded and $E_{\mathcal{U}}$ is reflexive. We point out that the reflexivity of E does not imply the reflexivity of $E_{\mathcal{U}}$ in general. The reflexive Banach spaces having one (and hence all) ultrapower(s) reflexive are called **superreflexive** and include, e.g., all L^p-spaces for

$1 < p < \infty$. For later use we recall that the dual of an ultrapower of a superreflexive Banach space is (isometric to) the ultrapower of the dual Banach space, *i.e.*, $(E_{\mathcal{U}})' = (E')_{\mathcal{U}}$ and the canonical bilinear form for elements $(x_n)_{n\in\mathbb{N}} \in E_{\mathcal{U}}$ and $(\varphi_n)_{n\in\mathbb{N}} \in (E')_{\mathcal{U}}$ is

$$\langle (x_n), (\varphi_n) \rangle = \mathcal{U} - \lim \langle x_n, \varphi_n \rangle.$$

This and other facts about superreflexive Banach spaces can be found in [He], §7, [H-M], §3, [Si], §10, §11 or [St]. We state the stability result deduced above from Theorem 1.5.

COROLLARY 2.3: *Let $T \in \mathcal{L}(E)$ be a power bounded operator on a superreflexive Banach space E. If*

$$(ABLP) \qquad\qquad \Lambda := \sigma(T) \cap \Gamma \quad \textit{is countable,}$$

then the extension $T_{\mathcal{U}}$ of T to every ultrapower $E_{\mathcal{U}}$ is stable.

Since the space $E_{\mathcal{U}}$ and the operator $T_{\mathcal{U}}$ are extensions of E and T respectively, it is clear that the stability of $T_{\mathcal{U}}$ is a much stronger property than just the stability of T. Therefore it might be worthwile to give it a name.

Definition 2.4: An operator $T \in \mathcal{L}(E)$ is called **superstable** if for every ultrapower $E_{\mathcal{U}}$ the extension $T_{\mathcal{U}} \in \mathcal{L}(E_{\mathcal{U}})$ of T is stable. ∎

With this terminology Corollary 2.3 says that power bounded operators on superreflexive Banach spaces satisfying the spectral condition $(ABLP)$ are superstable. These observations lead us to propose the following conjecture regarding Problem 1.8.

CONJECTURE 2.5: *A power bounded linear operator on a superreflexive Banach space is superstable if and only if it satisfies $(ABLP)$.*

Before proving this conjecture in the following section we present two examples showing that (a) superstability is strictly stronger than stability and (b) superreflexivity is necessary.

Example 2.6: (a) Let $E := l^2$ and $T \in \mathcal{L}(E)$ be the shift operator which clearly is stable (see Example 1.2 a)). Denote by e_n the canonical unit vectors in l^2 and take $\hat{e} := (e_n)_{n\in\mathbb{N}} \in E_{\mathcal{U}}$ for some ultrafilter $\mathcal{U}$. Since

$$\|T_{\mathcal{U}}^k \hat{e} - T_{\mathcal{U}}^l \hat{e}\| = \mathcal{U} - \lim \|e_{n+k} - e_{n+l}\| = \sqrt{2} \quad \text{for } k \neq l,$$

we see that $\{T_{\mathcal{U}}^n \hat{e} : n \in \mathbb{N}\}$ is not relatively compact, hence $T_{\mathcal{U}}$ is not stable.

(b) Let $E := l^p(l_n^\infty)$, $1 < p < \infty$, be the l^p-sum of the spaces l_n^∞. The space E is reflexive, but not superreflexive. Choose an increasing real sequence $0 < \alpha_n \uparrow 1$. The multiplication operators $T_n : (\xi_1, \dots, \xi_n) \mapsto (\alpha_1 \xi_1, \dots, \alpha_n \xi_n)$ on l_n^∞ induce an operator $T \in \mathcal{L}(E)$ by $T : (x_n) \mapsto (T_n x_n)$. This operator is a contraction having spectrum $\sigma(T) = \{\alpha_n : n \in \mathbb{N}\} \cup \{1\}$, hence it is stable. On the other hand the elements $z_m := (z_n^{(m)}) \in l^p(l_n^\infty)$ defined by $z_n^{(m)} = 0$ for $n \neq m$ and $z_m^{(m)} = (1, \dots, 1) \in l_m^\infty$ yield $\hat{z} := (z_m)_{m \in \mathbb{N}} \in E_{\mathcal{U}}$ for which $\{T_{\mathcal{U}}^n \hat{z} : n \in \mathbb{N}\}$ has no convergent subsequence, *i.e.*, $T_{\mathcal{U}}$ is not stable.

3. Spectral characterization of superstable operators

We now start to work on Conjecture 2.5 and, in the end, succeed by confirming it. To this end we recall or prove results from (i) the theory of ultrapowers, (ii) the geometry of Banach spaces and (iii) the theory of operators on Banach lattices which bare some interest in themselves. Their combination will then yield the desired and final theorem.

We always consider a power bounded operator $T \in \mathcal{L}(E)$ on some Banach space E. Since

$$|\!|\!| x |\!|\!| := \sup\{\|T^n x\| : n \in \mathbb{N}\}, \quad x \in E,$$

yields an equivalent norm on E making T a contraction we already assume $\|T\| \leq 1$.

In the first lemma we recall, from [St], Prop.2.1, that the formation of ultrapowers can be iterated.

Lemma 3.1: *Let E be a Banach space and let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on $\mathbb{N}$. Then the iterated ultrapower $(E_{\mathcal{U}})_{\mathcal{V}}$ is isometric to $E_{\mathcal{U} \times \mathcal{V}}$ where $\mathcal{U} \times \mathcal{V}$ is the ultrafilter on $\mathbb{N} \times \mathbb{N}$ defined by $W \in \mathcal{U} \times \mathcal{V}$ if and only if $\{n : \{m : (m,n) \in W\} \in \mathcal{U}\} \in \mathcal{V}$.*

The isometry $\Phi : (E_{\mathcal{U}})_{\mathcal{V}} \to E_{\mathcal{U} \times \mathcal{V}}$ is given by

$$\Phi(((x_{m,n})_{m \in \mathbb{N}} + c_{\mathcal{U}}(E))_{n \in \mathbb{N}} + c_{\mathcal{V}}(E_{\mathcal{U}})) := (x_{m,n})_{(m,n) \in \mathbb{N} \times \mathbb{N}} + c_{\mathcal{U} \times \mathcal{V}}(E).$$

Clearly $E_{\mathcal{U} \times \mathcal{V}}$ can be identified with an ultrapower $E_{\mathcal{W}}$ for an appropriate ultrafilter $\mathcal{W}$ on $\mathbb{N}$. Applying this to the ultrapower extensions of the operator $T \in \mathcal{L}(E)$ yields the following commuting diagram:

$$\begin{array}{ccc} E_{\mathcal{W}} & \xrightarrow{T_{\mathcal{W}}} & E_{\mathcal{W}} \\ \Phi \uparrow & & \uparrow \Phi \\ (E_{\mathcal{U}})_{\mathcal{V}} & \xrightarrow[(T_{\mathcal{U}})_{\mathcal{V}}]{} & (E_{\mathcal{U}})_{\mathcal{V}} \end{array}$$

Our next proposition is an immediate consequence of this observation.

PROPOSITION 3.2: *Let E be a Banach space and $T \in \mathcal{L}(E)$. Then T is superstable if and only if $T_{\mathcal{U}}$ is superstable for every ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*

By passing to an ultrapower it follows from Proposition 2.2 that the approximate point spectrum becomes point spectrum. But the boundary spectrum $\sigma(T) \cap \Gamma$ is always contained in $A\sigma(T)$. In view of Proposition 3.2 above this allows us to assume that

$$\Lambda := \sigma(T) \cap \Gamma$$

consists of eigenvalues only. We restrict T to the closed subspace generated by the corresponding eigenvectors and obtain an operator with 'discrete spectrum' (see [Sc], p.208 for this notion) which is always an invertible isometry (see Proposition 1.3). We will see that if such an operator has uncountably many eigenvalues it induces a unimodular multiplication operator M on a sufficiently large Banach sequence lattice. Here a **Banach sequence lattice** is a space Z of sequences which is a vector lattice with respect to the canonical ordering and is an ideal in l^∞ containing l^1. Moreover Z is complete with respect to a lattice norm.

PROPOSITION 3.3: *Let T be an isometry on a Banach space E and let $P\sigma(T)\cap\Gamma$ be uncountable. Then there is a Banach sequence lattice Z, an isometry $\Psi : Z \to E$, a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $P\sigma(T) \cap \Gamma$ and a multiplication operator $M \in \mathcal{L}(Z)$ given by $M(a_n) := (\lambda_n a_n)$ for $(a_n) \in Z$ such that $\sigma(M) = \overline{\{\lambda_n : n \in \mathbb{N}\}} \subseteq \Gamma$ is uncountable and the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{T} & E \\ \Psi \uparrow & & \uparrow \Psi \\ Z & \xrightarrow[M]{} & Z \end{array}$$

Proof: Via the map $\exp(2\pi i t) \mapsto t$, $t \in [0,1)$, we identify $P\sigma(T) \cap \Gamma$ with an uncountable subset D of $[0,1)$. Let $(\mu_\alpha)_{\alpha \in A}$ be a maximal $\mathbb{Q}$-linearly independent family in $D \setminus \mathbb{Q}$. Clearly $(\mu_\alpha)_{\alpha\in A}$ is uncountable. Let $(\mu_n)_{n\in N}$ be a countable subfamily of $(\mu_\alpha)_{\alpha\in A}$ such that the closures of $\{\mu_n : n \in \mathbb{N}\}$ and $\{\mu_\alpha : \alpha \in A\}$ coincide. Then $\lambda_n := \exp(2\pi i \mu_n) \in P\sigma(T) \cap \Gamma$ and $\overline{\{\lambda_n : n \in \mathbb{N}\}}$ is uncountable. The $\mathbb{Q}$-linear independence of $(\mu_n)_{n\in N}$ implies

$$\overline{\{(\lambda_1^m, \dots, \lambda_n^m) : m \in \mathbb{N}\}} = \Gamma^n \qquad \text{for every } n \in \mathbb{N} \tag{1}$$

(see [H-R], p.408).

Let now $x_n \in E$, $\|x_n\| = 1$, satisfy $Tx_n = \lambda_n x_n$, $n \in \mathbb{N}$. We claim that $(x_n)_{n\in N}$ is an unconditional basic sequence with unconditional constant 1 (see [L-T], p.18 for this notion).

In fact, for every $m, n \in \mathbb{N}$ and every choice of scalars $(a_1, \dots, a_n)$ we have

$$\|\sum_{k=1}^n a_k x_k\| = \|T^m(\sum_{k=1}^n a_k x_k)\| = \|\sum_{k=1}^n a_k \lambda_k^m x_k\|$$

since T is an isometry. By (1) this implies

$$\|\sum_{k=1}^n a_k x_k\| = \|\sum_{k=1}^n a_k \beta_k x_k\| \tag{2}$$

for all choices of β_k, $|\beta_k| = 1$, $1 \le k \le n$. Then for all $m, n \in \mathbb{N}$, $n < m$, and every sequence of scalars $(a_k)_{k\in N}$ we have

$$\|\sum_{k=1}^n a_k x_k\| \le \frac{1}{2}(\|\sum_{k=1}^m a_k x_k\| + \|\sum_{k=1}^n a_k x_k - \sum_{k=n+1}^m a_k x_k\|) = \|\sum_{k=1}^m a_k x_k\|,$$

hence $(x_n)_{n\in N}$ is a Schauder basis of $G := \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$ (cf. [L-T], 1.a.3). Moreover, if $\sum_{k\in N} a_k x_k$ converges, then by (2) the series $\sum_{k\in N} a_k \beta_k x_k$ is convergent for every sequence $(\beta_n)_{n\in N}$ of unimodular scalars and

$$\|\sum_{k\in N} a_k x_k\| = \|\sum_{k\in N} a_k \beta_k x_k\|. \tag{3}$$

This shows in fact that $(x_n)_{n\in N}$ is an unconditional basis with unconditional constant 1 (cf. [L-T], 1.c.5, 1.c.1).

If $\sum_{k\in\mathbb{N}} a_k x_k$ is convergent, then the series $\sum_{k\in\mathbb{N}} b_k x_k$ is convergent for every sequence of scalars $(b_k)_{k\in\mathbb{N}}$ with $|b_k| \leq |a_k|$, $k \in \mathbb{N}$ (cf. [L-T], 1.c.6) and by (3) and [L-T], 1.c.7, we have

$$(4) \qquad \Big\|\sum_{k\in\mathbb{N}} b_k x_k\Big\| \leq \Big\|\sum_{k\in\mathbb{N}} a_k x_k\Big\| = \Big\|\sum_{k\in\mathbb{N}} |a_k| x_k\Big\|.$$

Hence $Z := \{(a_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sum_{k\in\mathbb{N}} a_k x_k \text{ converges}\}$ endowed with the norm $\|(a_k)_{k\in\mathbb{N}}\| := \|\sum_{k\in\mathbb{N}} |a_k| x_k\|$ is a Banach sequence lattice and the mapping

$$\Psi : (a_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} a_k x_k$$

is an isometry. Clearly, if $M : Z \to Z : (a_k)_{k\in\mathbb{N}} \mapsto (\lambda_k a_k)_{k\in\mathbb{N}}$ is the multiplication by $(\lambda_k)_{k\in\mathbb{N}}$ then $\sigma(M) = \overline{\{\lambda_n : n \in \mathbb{N}\}} \subseteq \Gamma$ is uncountable and the diagram above commutes. ∎

Remark: If E is reflexive, then the dual space Z' of Z is again a Banach sequence lattice and the duality is given by

$$\langle (a_k), (b_k)\rangle := \sum_{k\in\mathbb{N}} a_k b_k$$

for $(a_k) \in Z$, $(b_k) \in Z'$. ∎

In the third preparatory proposition we show that ultrapowers of operators as they appeared in Proposition 3.3 always contain multiplication operators on a diffuse Banach lattice. Here we follow the terminology and use various results from [Sc], Ch.II. But first we need the following decomposition of probability vectors in finite dimensional Banach lattices.

LEMMA 3.4: *Let $E := \mathbb{C}^n$ be endowed with a lattice norm $\|.\|$ and its dual norm $\|.\|'$. Then for every probability vector $\alpha = (\alpha_1, \dots, \alpha_n)$, i.e., $\alpha \in \mathbb{R}^n_+$ and $\sum_{k=1}^n \alpha_k = 1$, there exist $\beta = (\beta_1, \dots, \beta_n) \in E_+$ and $\gamma = (\gamma_1, \dots, \gamma_n) \in E'_+$ such that $\|\beta\| = 1 = \|\gamma\|'$ and $\beta_k \gamma_k = \alpha_k$ for $1 \leq k \leq n$.*

This lemma is proved, e.g., in [T-J], Lemma 39.3, and allows one to embed 'diffuse multiplication operators' into the ultrapower of operators as in Proposition 3.3. More precisely the following holds.

PROPOSITION 3.5: *Let E be a superreflexive Banach sequence lattice and $T \in \mathcal{L}(E)$ an isometric multiplication operator $T(x_n) := (\lambda_n x_n)$ for $(x_n)_{n\in\mathbb{N}} \in E$ with $|\lambda_n| = 1$ for all $n \in \mathbb{N}$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. If $\Lambda := \overline{\{\lambda_n : n \in \mathbb{N}\}}$ is uncountable, then there exists a diffuse probability measure μ supported on Λ, a Banach function lattice W with continuous injections*

$$L^\infty(\Lambda, \mu) \hookrightarrow W \hookrightarrow L^1(\Lambda, \mu)$$

and a positive isometry $P : W \to E_\mathcal{U}$ such that the following diagram commutes:

$$\begin{array}{ccc} E_\mathcal{U} & \xrightarrow{\;T_\mathcal{U}\;} & E_\mathcal{U} \\ P \uparrow & & \uparrow P \\ W & \xrightarrow[\;M\;]{} & W \end{array}$$

Here, M is the multiplication operator $Mf(\lambda) := \lambda \cdot f(\lambda)$ for every $f \in W$, $\lambda \in \Lambda$.

Proof: Since E is superreflexive we observe that the dual of $E_\mathcal{U}$ can be identified with $E'_\mathcal{U} = (E')_\mathcal{U}$ with respect to the bilinear form

$$\langle (u_n), (\varphi_n) \rangle := \mathcal{U} - \lim \langle u_n, \varphi_n \rangle$$

for $(u_n) \in E_\mathcal{U}$, $(\varphi_n) \in E'_\mathcal{U}$ (see the remarks preceding Corollary 2.3). Next the existence of a diffuse probability measure $\mu \in C(\Lambda)'$ follows from [Se], 19.7.6, 8.5.5. Since the atomic measures are weak*-dense in $C(\Lambda)'$ we find for every $n \in \mathbb{N}$ a probability vector $(\alpha_{1,n}, \dots, \alpha_{n,n})$ such that the sequence of atomic measures

$$\mu_n := \sum_{k=1}^{n} \alpha_{k,n} \delta_{\lambda_k}$$

weak*-converges to μ. Consider then the n-dimensional subspace

$$E_n := \operatorname{lin}\{e_1, \dots, e_n\}$$

in E and its dual $E'_n := \operatorname{lin}\{e_1, \dots, e_n\}$ in E' where e_k are the canonical basis vectors of the Banach sequence lattices E and E'. By Lemma 3.4 there exist

$$u_n := \sum_{k=1}^{n} \beta_{k,n} e_k \in E_n \quad \text{and} \quad \varphi_n := \sum_{k=1}^{n} \gamma_{k,n} e_k \in E'_n$$

such that $\|u_n\| = 1 = \|\varphi_n\|'$, $\beta_{k,n}\gamma_{k,n} = \alpha_{k,n}$ for $1 \leq k \leq n$ and therefore

$$\langle u_n, \varphi_n \rangle = \sum_{k=1}^{n} \beta_{k,n}\gamma_{k,n} = \sum_{k=1}^{n} \alpha_{k,n} = 1 \quad \text{for all } n \in \mathbb{N}.$$

Using the elements $u_n \in E_+$ and $\varphi_n \in E'_+$ we define elements in the ultrapower by

$$\hat{u} := (u_n) \in E_{\mathcal{U}} \quad \text{and} \quad \hat{\varphi} := (\varphi_n) \in E'_{\mathcal{U}}.$$

The ultrapower $E_{\mathcal{U}}$ is a reflexive Banach lattice and therefore has order continuous norm (see [Sc], II.5.10). Hence there exists a band decomposition

$$E_{\mathcal{U}} = E_1 \oplus E_2,$$

such that $\hat{\varphi}$ is strictly positive on E_1 while $\hat{\varphi}$ vanishes on E_2. By [Sc], IV.9.3, we conclude that E_1 is a dense ideal in the AL-space $(E_{\mathcal{U}}, \hat{\varphi})$ (see [Sc], II.8.Example 1 for this notion). Define now a linear operator P from $C(\Lambda)$ into the principal ideal $(E_{\mathcal{U}})_{\hat{u}}$ generated by $\hat{u}$ ([Sc], p.57) by setting

$$Pf := \Big(\sum_{k=1}^{n} \beta_{k,n} f(\lambda_k) e_k\Big)_{n\in\mathbb{N}}$$

for every $f \in C(\Lambda)$. This operator is positive and satisfies

$$\begin{aligned}
\langle |Pf|, \hat{\varphi} \rangle &= \mathcal{U} - \lim \Big\langle \Big|\sum_{k=1}^{n} \beta_{k,n} f(\lambda_k) e_k\Big|, \sum_{k=1}^{n} \gamma_{k,n} e_k \Big\rangle \\
&= \mathcal{U} - \lim \Big\langle \sum_{k=1}^{n} \beta_{k,n} |f(\lambda_k)| e_k, \sum_{k=1}^{n} \gamma_{k,n} e_k \Big\rangle \\
&= \mathcal{U} - \lim \sum_{k=1}^{n} \beta_{k,n}\gamma_{k,n} |f(\lambda_k)| \langle e_k, e_k \rangle \\
&= \mathcal{U} - \lim \sum_{k=1}^{n} \alpha_{k,n} |f(\lambda_k)| \\
&= \mathcal{U} - \lim \langle |f|, \mu_n \rangle \\
&= \langle |f|, \mu \rangle
\end{aligned}$$

for the measures μ and μ_n fixed above. Hence P extends continuously to an isometry from $L^1(\Lambda, \mu)$ into $(E_{\mathcal{U}}, \hat{\varphi}) = (E_1, \hat{\varphi})$ mapping $L^\infty(\Lambda, \mu)$ into the principal ideal in E_1 generated by the component of $\hat{u}$ in E_1. If we denote by

W the inverse image of E_1 we obtain a Banach function lattice and injections $L^\infty(\Lambda,\mu) \hookrightarrow W \hookrightarrow L^1(\Lambda,\mu)$. In the final step we take the multiplication operator

$$Mf(\lambda) := \lambda \cdot f(\lambda)$$

defined on $L^1(\Lambda,\mu)$. For $f \in C(\Lambda)$ it follows that

$$\begin{aligned} PMf &= (\sum_{k=1}^{n} \beta_{k,n}\lambda_k f(\lambda_k)e_k)_{n\in\mathbb{N}} \\ &= (T(\sum_{k=1}^{n} \beta_{k,n} f(\lambda_k)e_k))_{n\in\mathbb{N}} \\ &= T_{\mathcal{U}}(\sum_{k=1}^{n} \beta_{k,n} f(\lambda_k)e_k)_{n\in\mathbb{N}} \\ &= T_{\mathcal{U}}Pf. \end{aligned}$$

By continuous extension the same holds for every $f \in W$, hence the stated diagram commutes. ∎

The remarkable aspect of the above multiplication operator $M \in \mathcal{L}(W)$ is twofold.

1. The operator M is an isometry, hence the stable subspace

$$W_s := \{f \in W : \lim_{n\to\infty} \|M^n f\| = 0\}$$

reduces to zero.

2. The operator M has empty point spectrum, hence

$$W_r := \overline{\text{lin}}\{f \in W : Mf = \lambda f,\, |\lambda| = 1\}$$

reduces to zero.

Such behavior has already been encountered in Example 1.7 and implies that M and hence $T_{\mathcal{U}}$ is unstable. As a consequence we have the following

COROLLARY 3.6: *If $T \in \mathcal{L}(E)$ is as in Proposition 3.5 and has uncountable spectrum $\sigma(T) = \Lambda := \overline{\{\lambda_n : n \in \mathbb{N}\}}$, then T is not superstable.*

We now tie together Propositions 3.2, 3.3 and 3.5 in order to prove Conjecture 2.5. We have to show that given a contraction $T \in \mathcal{L}(E)$, E superreflexive, for which

$$\Lambda := \sigma(T) \cap \Gamma \text{ is uncountable,}$$

we can find an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and a subspace of $E_{\mathcal{U}}$ on which $T_{\mathcal{U}}$ is not stable. First, Proposition 3.2 allows to assume that Λ is contained in the point spectrum. Take $\lambda_n \in P\sigma(T)$ such that $\Lambda = \overline{\{\lambda_n : n \in \mathbb{N}\}}$ and eigenvectors $x_n \in E$ such that $Tx_n = \lambda_n x_n$ and consider the closed, T-invariant subspace F generated by $\{x_n : n \in \mathbb{N}\}$. The restriction of T to F has discrete spectrum and we can apply Proposition 3.3 obtaining a Banach sequence lattice Z on which T acts as the multiplication operator M corresponding to the values $\{\lambda_n : n \in \mathbb{N}\}$. The ultrapower of $M \in \mathcal{L}(Z)$ is contained in $T_{\mathcal{U}} \in \mathcal{L}(E_{\mathcal{U}})$ and contains an unstable part according to Proposition 3.5 and Corollary 3.6. Hence T is not superstable and we have proved the final result.

THEOREM 3.7: *Let T be a power bounded linear operator on a superreflexive Banach space E. Then*

$$(ABLP) \qquad \Lambda := \sigma(T) \cap \Gamma \text{ is countable}$$

if and only if T is superstable.

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